

THE ONSET OF NATURAL CONVECTION FROM TIME-DEPENDENT PROFILES

IAN F. DAVENPORT and C. JUDSON KING

Department of Chemical Engineering, and Lawrence Berkeley Laboratory, University of California, Berkeley, California 94720, U.S.A.

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Abstract—Experiments were conducted on the initiation of natural convection in deep pools (penetration depth is small compared with the fluid depth). Fluids covering a wide range of Prandtl number were heated from below. A constant value of Rayleigh number based upon density boundary layer thickness (Ra_ρ) correlates the effects of surface boundary condition and fluid depth. These effects and those of Prandtl number and wall spacing can be interpreted in terms of growth rate-amplification theory; however a constant value of the Péclet number at the onset of convection was found to correlate the data better than did a constant value of the amplification factor. Theoretical reasons for this result are presented.

NOMENCLATURE

C_p , heat capacity at constant pressure [cal/g°C];
 D , fluid width [cm];
 g , gravity constant [cm/s²];
 H , fluid depth [cm];
 k , thermal conductivity [cal/cm s°C];
 l , density boundary layer thickness using the segment approximation
 $= 2 \int_0^\infty \left(\frac{T - T_b}{T_s - T_b} \right) dz$ [cm];
 Pr , Prandtl number = $C_p \mu / k$;
 q , vertical heat flux;
 Ra_H , Rayleigh number based on fluid depth
 $= \frac{\Delta \rho g H^3}{\mu \alpha}$;
 Ra_ρ , Rayleigh number based on effective density boundary layer thickness
 $= \frac{\Delta \rho g \alpha^{1/3} l^3}{\mu \alpha}$;
 Ra_c , Rayleigh number based on conduction time
 $= \frac{\Delta \rho g \alpha^{1/3} t^3}{\mu}$;
 S , surface temperature shape factor;
 Sc , Schmidt number = $\mu / \rho D$;
 t , conduction time prior to convection onset [s];
 T , temperature [°C];
 ΔT , temperature difference between bulk and surface conditions;

u , representative convection velocity;
 w , amplification factor;
 z , vertical coordinate [cm].

Greek letters

α , thermal diffusivity [cm²/s];
 β , coefficient of thermal expansion [°C⁻¹];
 μ , viscosity [p];
 ρ , density [g/cm³];
 $\Delta \rho$, difference in density between surface liquid and bulk liquid [g/cm³].

Subscripts

b , bulk;
 s , surface;
 c , value at observable convection.

INTRODUCTION

THE RELIABLE prediction of conditions at the onset of convection from time-dependent driving forces is of importance to the engineer in designing commercial equipment and to the mathematician and physicist in understanding natural phenomena. The results of linear stability theory in this area [3] have generally predicted critical Rayleigh numbers which have been approximately two orders of magnitude below those observed. The apparent reason for this discrepancy is that linear stability theory predicts the condition at which an infinitesimal disturbance will begin to grow in amplitude, whereas convection is first observed when the amplitude has reached such a finite size as to produce a "measurable" influence on the transport process. A growth period is required, as it has been suggested [5], for the necessary amplification

of the convection motion, and, in a situation where the driving force for convection is changing with time, such as diffusion into a semi-infinite medium, the net result is that convection is first observed at a much higher Rayleigh number than that corresponding to first growth.

Although this mechanistic picture appears reasonable, especially in light of the advanced work of Gebhart and co-workers [7] in a similar study of vertical wall natural convection, the means to predict the first observable convection is clouded; i.e. reported incipient convection conditions occur over a range of amplification factor from 10^1 to 10^6 [4], and theoretical predictions are available only for a linear increase over time or a step change in surface temperature. The purpose of this paper is to present an analysis aimed at improving the means of predicting the onset of convection from time-dependent driving forces at solid-liquid surfaces.

THEORY

The stability of a non-linear density profile $\rho(z)$ can be defined [3] in terms of a Rayleigh number, Ra_p , by assuming the profile to consist of two linear segments, where

$$l = \int_0^{\infty} \frac{\rho(z)}{\rho(0)} dz \quad (1)$$

and

$$Ra_l = \frac{\rho g \beta (T_s - T_B) l^3}{\mu \alpha} \quad (2)$$

For a time-dependent profile,

$$l = K\sqrt{(\alpha t)} \quad (3)$$

The constant of proportionality (K) in equation (3) will be dependent upon the shape of the profile, which in turn will depend upon the behavior of the surface temperature (T_s) with time. One means of identifying the shape of the density curve is through

$$S = \frac{\int_0^t T_s(\theta) d\theta}{T_s t} \quad (4)$$

S is 1.0 for a step change and 0.5 for a linear change in surface temperature with time. This is the simplest parameter that can describe the surface temperature curve. It has the drawback that several curve shapes can be described by a given value of S . The relationship between K and S was found by assuming $T_s = t^n$.

The time-dependent Rayleigh number is found from equations (2) and (3):

$$Ra_l = \frac{\rho g \beta (T_s - T_B) \alpha^{\frac{1}{2}} t^{\frac{3}{2}}}{\mu} \quad (5)$$

EXPERIMENTATION

(1) Effect of Prandtl number on Ra_l

The apparatus to measure Ra_l , and its variation with Prandtl number, was designed to produce a vertical, one-dimensional temperature profile in the liquid so as to conform strictly with the theoretical models. A cylinder (9 cm i.d., 4.5 cm height), made of Plexiglass which has similar thermal properties as the liquids tested, was attached to a variable 750 W plate heater. The surface of the heater was carefully designed to be isothermal. The assembly was fitted to a leveling table so that the heating surface was horizontal. A 1 mil copper-constantan thermocouple was placed on the surface of the isothermal heating plate, being insulated from it by a thin sheet of mica. The temperature response was recorded over ranges of 50 μ V–2 mV.

A run was begun by allowing the test liquid to come to thermal equilibrium with the surroundings, as indicated by the change in thermocouple response. A d.c. heating power supply was adjusted to give an initial high power input, followed by a constant power level to produce a linear increase of the heating surface temperature with time. When the test fluid was a liquid, convection initiation was observed by the first deviation from linearity of the surface temperature with time. However, with air, a thin thermocouple was suspended about 1 mm above the surface, and this detected the onset of convection much better than the thermocouple on the air-solid surface.

The variation in Prandtl number came from testing fluids which varied in viscosity. These fluids were air ($Pr = 0.7$), methanol ($Pr = 7.7$), n-decane ($Pr = 13.3$), n-undecane ($Pr = 17$), n-butanol ($Pr = 43$), n-hexanol ($Pr = 64$), n-octanol ($Pr = 108$), silicone oils 50 cs ($Pr = 450$), 100 cs ($Pr = 890$) and 1000 cs ($Pr = 8500$).

For a particular fluid, the conditions at the onset of convection were identified by the temperature elevation, ΔT_c , and the period of conduction, t_c , prior to observable convection. These critical conditions, where ΔT_c for a fluid varied by almost a factor of 10, showed that ΔT_c varied as $t_c^{-\frac{1}{2}}$ for each fluid, as required for correlation by the time-dependent Rayleigh number. The experimental results are plotted in Fig. 1 and tabulated elsewhere [4].

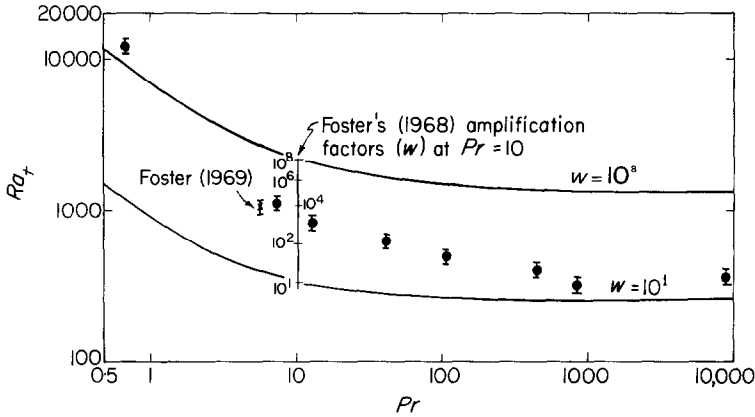


FIG. 1. Experimental values of Ra_+ vs Pr for a linear temperature decay at a fixed surface in a deep pool. ● Experimental data.— Foster's (1968) theoretical curve for a constant amplification factor of 10^1 and 10^8 .

Predicted relationships between Ra_+ and Pr calculated by Foster [5] using Amplification Theory with constant amplification factors of 10^1 and 10^8 are also given in Fig. 1. The data show that the apparent amplification factor corresponding to onset decreases as Pr increases. Foster [6] published experimental results for heating water in a similar apparatus to that used in this report. With Pr of 6.6, he correlated his results with an amplification factor of 10^3 – 10^5 , which agrees well with the present results.

Onat and Grigull [9] report values of Ra_+ which are lower by approximately a factor of five than the present results, over a range of Pr from 7 to 4400. Their experiments were carried out in a similar manner to the present experiments and those of Foster [6]. The lower values of Ra_+ could be due to high residual velocity resulting from a short period of rest between runs. Such an effect was noted in our preliminary investigations. In the current experiments the liquid was kept as isothermal and motionless as possible as a starting condition, in an attempt to produce reproducible and low amplitude pre-

convection fluid disturbances. Consequently, rest periods between runs were up to $1\frac{1}{2}$ h for a run lasting 20–150 s. Runs with longer conduction times before instability were left longer to rest between runs.

(2) Effect of wall spacing on Ra_+

As the ratio of fluid width (D) to depth (H) decreases, an influence of the walls on the incipient convection can be expected.

Experiments in the present study observed the effect on Ra_+ of D^2/at , the ratio of the square of the fluid width to the square of the penetration depth. Data were obtained by temporarily placing a Plexiglas block with a well finished face on the heating surface of the large vessel used for the Ra_+ vs Pr experiments. Holes of 5.7 mm and 8.7 mm were drilled into the block to contain the fluid. The results are shown in Fig. 2. It can be seen that wall effects are negligible for n-octanol ($Pr = 108$) and n-decane ($Pr = 13$), provided $at/D^2 < 0.1$. For greater values of at/D^2 , Ra_+ rises sharply.

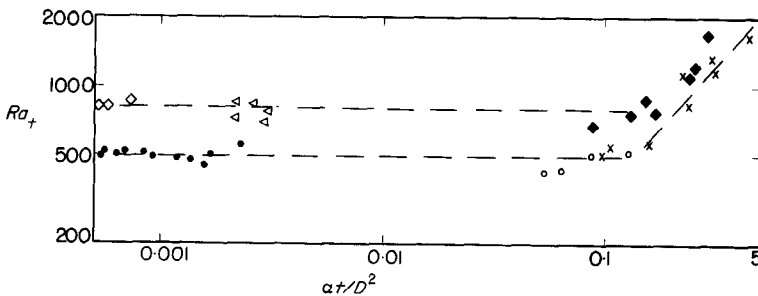


FIG. 2. Experimental values of Ra_+ vs width factor (at/D^2) for the fixed-surface, deep-pool case. n-Octanol: ○ $D = 8.9$ cm, ● $D = 0.87$ cm, × $D = 0.57$ cm. n-Decane: ◇ $D = 8.9$ cm, <math>D = 5.9 cm, ◆ $D = 0.57$ cm.

(3) *Effect of a vibrating heating surface on Ra_i*

Experimental data were obtained by mounting the apparatus used to determine Ra_i vs Pr on top of a vertical sinusoidal vibrator described elsewhere [4]. The frequency (f) range was 10–10000 cps, and the amplitude (A) range was 10^{-7} – 10^{-1} in., but these variables were limited in their independence so that A^2f^3 varied by approximately 1000 at most. Different bulk fluid motions were produced by the upper surface of the liquid being either flush with the vessel lid or free of it. The two different wave forms produced in the liquid with free and fixed upper surfaces did not produce results which were significantly different from each other. The exception came at low frequencies and high amplitudes, where the liquid surface was broken for the free surface case, which lowered Ra_i significantly.

The results tabulated in [4] showed that, as the forcing vibration decreased in frequency and increased in amplitude, the onset of convection occurred earlier. The greatest destabilizing effect occurred at approximately 60 cps and 0.01 in. amplitude, where the value of Ra_i was lowered by 40 per cent from the value obtained under "vibration-free" condition discussed in Section (1) above.

(4) *Effect of fluid depth on Ra_i*

The transition from the deep-pool (semi-infinite medium) to the shallow-pool behavior was observed experimentally by varying the fluid depth and the

heating rate. The fluid depth was regulated by supporting an aluminum cylinder, 5 cm high and slightly less than 9 cm wide, with its base parallel to the heating surface on a Plexiglas annulus of known height attached to the heating surface. Aluminum was selected to maintain the temperature of the upper surface of the liquid constant. A thermocouple attached to the base of the metal cylinder showed that the upper liquid surface temperature varied by less than 5 per cent of the total temperature drop across the fluid.

The correlating parameter to show the change in critical Rayleigh number, based on either fluid depth, Ra_H , or penetration depth, Ra_i , at shallow depths and constant Pr is $\alpha t/H^2$. The results have been plotted on Fig. 3 for three fluids and for liquid depths as small as 2.05 mm. The transition from the deep-pool mechanism to the shallow-pool mechanism is clearly seen in Fig. 3. The shallow-pool data converge on an asymptote which is independent of Pr and heating rate and dependent on fluid depth ($Ra_H = 1700$). The deep pool asymptotes are independent of H but are dependent on Pr , as reported in Fig. 1.

There are no previously published shallow-pool experimental results for constant temperature at the upper surface. However, some shallow-pool experimental results for an adiabatic upper surface have been reported by Soberman [10]. His values of Ra_H are below the current results and lower than what was theoretically predicted by Currie [3] using linear stability analysis. Foster [6] reported values for

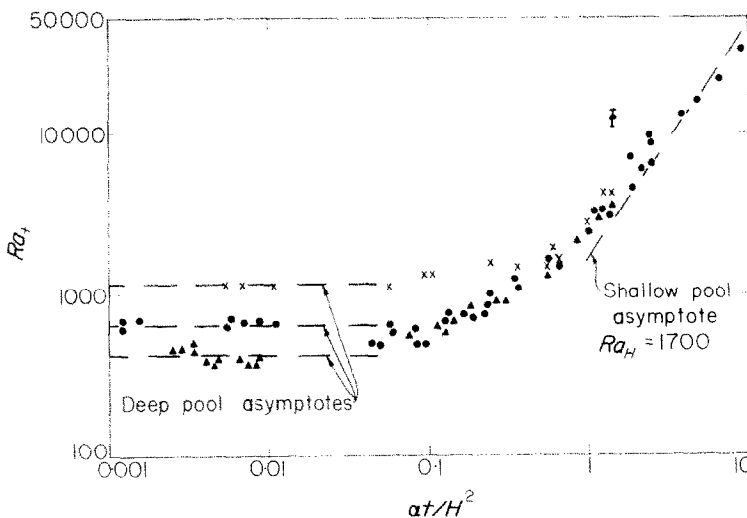


FIG. 3. Experimental values of Ra_i vs depth factor ($\alpha t/H^2$) showing the transition from deep- to shallow-pool conditions for the fixed-surface case. \times Methanol, $Pr = 7.6$; \bullet butanol, $Pr = 43$; \blacktriangle silicone oil 50 cs, $Pr = 450$; \bullet Foster's (1969) datum point for water.

convection onset in shallow pools with a linear increase in the lower surface temperature with time. The liquid at the upper surface was open to the air and so was closer to an insulated surface than to an isothermal surface. The critical Rayleigh numbers from Foster are estimated as $Ra_c = 10800-12700$ at $\alpha t/H^2 = 1.8$, as shown in Fig. 3.

(5) *Effect of the shape of the density profile in the fluid on Ra_c*

The apparatus used to investigate the influence of density profile shape on Ra_c was similar to that used to determine Ra_c vs Pr , with the exception that the heating at the surface in this case was produced by a thermoelectric heat pump rather than by a plate heater. The variation of the heated surface temperature with time was controlled by monitoring the current to the heat pump. In this way, curve shapes ranging from a step function to a linear time increase were produced.

The results for n-octanol have been plotted in Fig. 4. They show that Ra_c decreases as S increases in the range $0.5 < S < 0.9$. Similar dependence of Ra_c on S has been found for n-undecane, n-butanol, n-hexanol, and n-octanol at a free surface; these results have been reported by Davenport and King [4].

DISCUSSION

Previously published convection initiation data from time-dependent density profiles [1, 2, 6, 8, 11] have been reported in the form of Ra_c , except for Spangenberg and Rowland [11], who employed a value of Rayleigh number based on penetration depth somewhat similar to equation (2). The disadvantage of Ra_c as a correlating parameter is that its value is dependent upon the thermal boundary conditions of the experiment. A more general presentation is in

the form of Ra_p , which overcomes this limitation. Consequently, the Ra_c data in Figs. 3 and 4 have been transformed to the Ra_p form, using equations (1)-(3).

The lack of dependence of Ra_p upon the density profile shape (S) can be seen in Fig. 5. The scatter in Ra_p at high values of S is probably due to the assumed relationship between the length ratio (K)

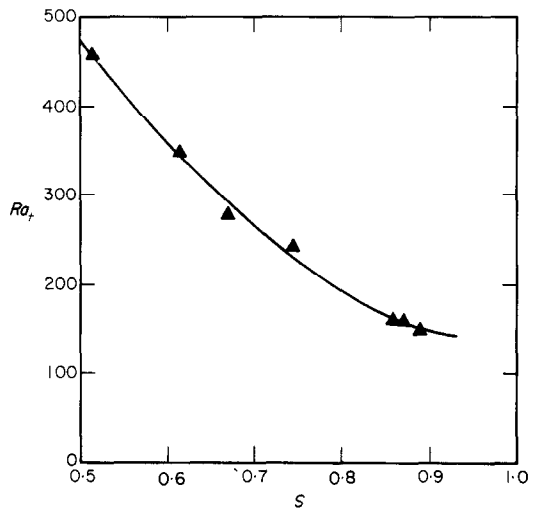


FIG. 4. Ra_c vs shape factor (S) for n-octanol in a deep pool.

and the shape factor (S), mentioned earlier. The weakness of the general application of the relationship stems from one value of S describing many density shape profiles. Notwithstanding this weakness, Fig. 5 shows S to be adequate as an approximation, and it is easy to compute.

The difference between conditions at observed incipient convection and at first growth can be seen in Fig. 6, where the linear stability results of Currie [3], applicable to this physical situation, have been plotted to show the conditions at first "possible"

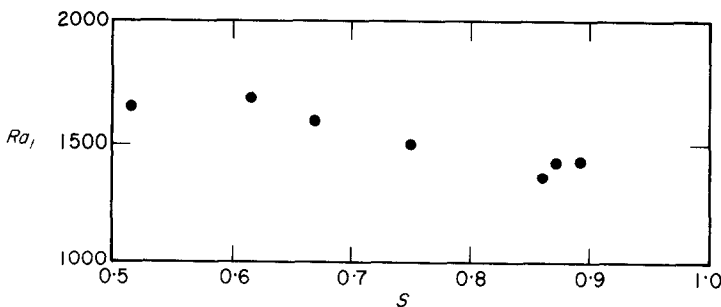


FIG. 5. Experimental values of Ra_p for n-Octanol ($Pr = 108$).

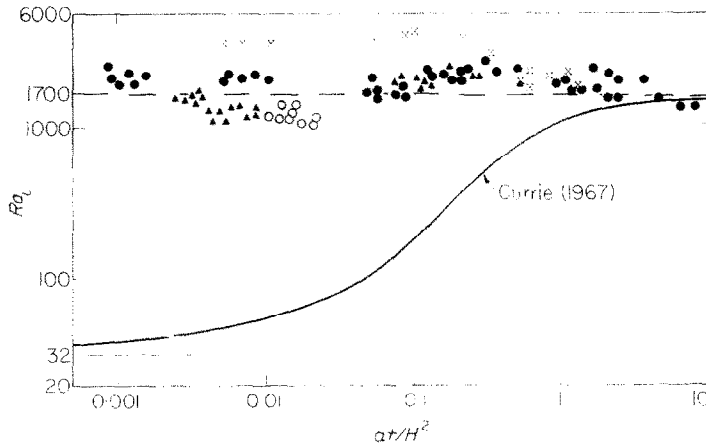


FIG. 6. Experimental values of Ra_i as a function of the depth factor ($\alpha t/H^2$). ●-n-Butanol, $Pr = 43$; ▲silicone oil 50 cs, $Pr = 465$; ○silicone oil 1000 cs, $Pr = 8500$; × methanol, $Pr = 7.6$.

convection motion. The apparent reason for this difference is slow growth rates of the convection motion until Ra_i is of the order of 1000, whereupon the growth rate increases rapidly. The sudden increase in growth rate can be seen from the velocity amplification ratios calculated by Foster [6] as a function of Ra_i for a linear change in surface temperature with time and plotted on Fig. 1. In this case, Ra_i is 3.4 times Ra_c . For the high Pr limit, the average convection velocity is predicted from Fig. 1 to be amplified by a factor of 10 for Ra_i between 0 and 850, while the second and third decade amplifications occur for Ra_i between 850 to 1550, and 1550 to 2100 respectively. Although the amplification time will be dependent upon the initial velocity distribution in the pre-convective fluid, the growth rate is so steep around Ra_i of 1500 (depending upon the Prandtl number) that experimentally observed Rayleigh numbers fall within a narrow range. Obviously this is of great computational value to the design engineer.

The increase in Ra_i (and Ra_c) with decreasing Prandtl number for deep pool conditions as shown in Fig. 1 is qualitatively in agreement with Foster's [6] theoretical predictions. However, the amplification factor corresponding to observable convection appears to increase as the Prandtl number decreases. Two explanations for this seem possible.

(1) As viscosity and Prandtl number decrease, the magnitude of the convective velocity at the onset of observable convection increases.

(2) The motion in the viscous fluid before heating is begun is greater than in the non-viscous fluid. Both (1) and (2) would account for a lower amplification factor for observable convection.

The importance of point (1) above can be seen when the components of the actual heat flux are analyzed: i.e.

$$q = -\alpha \frac{\partial(\rho C_p T)}{\partial z} + u(\rho C_p T). \quad (6)$$

The onset of convection will be observed when the convection flux is first comparable to the conduction flux; i.e. the Péclet number

$$\frac{u_c l_c}{\alpha} \quad \text{or} \quad \frac{u_c t_c^{\frac{1}{2}}}{\alpha^{\frac{1}{2}}}$$

is $O(1)$ where u_c and l_c are convection velocity and penetration depth respectively at the onset of observable convection, and α is the thermal diffusivity. In the present experiments, it was observed that, as the Prandtl number decreased, the critical penetration depth generally decreased, as manifested by shorter conduction periods (t_c). N-decane ($Pr = 13.7$) showed convection in the range of 26–118 s, while 1000 cs silicone oil ($Pr = 8500$) showed convection in the range of 276–617 s. The thermal diffusivity of liquids tested did not vary appreciably from 10^{-3} cm²/s. If the onset of convection occurs at a constant value of the Péclet number, the convection velocity u_c required should increase as the penetration depth l_c decreases. Indeed, a closer inspection of the data showed that Ra_i was up to 20 per cent higher for the shortest conduction periods (t_c) of each fluid tested. Interestingly, Foster's [6] datum point for water which was at much longer conduction times than the organic fluids with similar Prandtl number in the present experimental results is at a slightly lower amplification factor, as would be expected if a lower u_c is required.

The influence of the pre-convection fluid motion on Ra_1 was illustrated with the vibration results. Intense fluid vibration only decreased Ra_1 to a value around 1000. Here, once again, the results show that a major portion of the amplification time is spent during a low growth rate period and that these growth rates increase rapidly as Ra_1 approaches 1000. The vibration either provided a larger initial velocity disturbance or a faster amplification process. Unfortunately, not enough data are available to distinguish which factor dominated.

The effect of fluid depth on Ra_1 can be seen from Fig. 6. The amplification process appears to be virtually independent of the fluid depth for high Prandtl number fluids, whereas for low Prandtl number fluids, Ra_1 decreases as the penetration depth approaches the fluid depth. Both the thermal and momentum boundary layers in the low Prandtl number fluid are being confined to the fluid depth at these shallow pool conditions which removes the dependence of the amplification process on the Prandtl number. The behavior is then similar to a high Prandtl number fluid.

Finally, the results of the narrow fluid width and depth experiments show that the amplification process considers the fluid as semi-infinite, as shown by the constant value of Ra_1 for a given fluid, provided the dimensions of the fluid are greater than approximately twice the penetration depth at the onset of convection.

CONCLUSIONS

The analysis has shown that Ra_1 corresponding to first observable convection occurs at $O(10^3)$ for high Prandtl number fluids. This observation is consistent with the amplification model, which predicts that growth rates should increase rapidly as Ra_1 approaches $O(10^3)$ for high Prandtl number fluids. However, a physical interpretation of the results showed that although the amplification factor at

observable convection can vary by orders of magnitude, a constant value of the Péclet number

$$\frac{u_c t_c^{1/2}}{\alpha^{1/2}}$$

interprets onset conditions over a wide range of conditions.

As the Prandtl number decreases, the experimentally observed increase in the critical value of Ra_1 is in qualitative agreement with the amplification model predictions.

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L'ÉTABLISSEMENT DE LA CONVECTION NATURELLE A PARTIR DE PROFILS DÉPENDANT DU TEMPS

Résumé—Des expériences ont été conduites pour étudier l'initiation de la convection naturelle dans des réservoirs profonds (la profondeur de pénétration est petite comparée à celle du fluide). Des fluides couvrant une large gamme de nombre de Prandtl étaient chauffés par le bas. Une valeur constante du nombre de Rayleigh basé sur l'épaisseur de couche limite de densité (Ra_1) rassemble les effets de condition aux limites sur la surface et de profondeur du fluide. Ces effets, ceux du nombre de Prandtl et ceux de l'espacement des parois peuvent être interprétés en termes de vitesse de croissance dans une théorie de l'amplification. Néanmoins une valeur constante du nombre de Péclet au déclenchement de la convection unifie les résultats mieux que ne le ferait une valeur constante du facteur d'amplification. On présente des justifications théoriques de ce résultat.

DAS EINSETZEN VON FREIER KONVEKTION AN ZEITABHÄNGIGEN PROFILLEN

Zusammenfassung—Es wurden Versuche durchgeführt über das Einsetzen von freier Konvektion in tiefen Becken (Eindringtiefe klein im Vergleich zur Fluidtiefe). Flüssigkeiten in einem weiten Prandtl-Zahlbereich wurden von unten beheizt. Ein konstanter Wert der Rayleigh-Zahl, gebildet mit der Dichtegrenzschichtdicke (Ra_i) korreliert die Einflüsse von Oberflächenrandbedingung und Fluidtiefe. Diese Effekte und solche von Prandtl-Zahl und Wandabstand können interpretiert werden mit Hilfe der Theorie der Verstärkung der Wachstumsrate; es zeigte sich jedoch, dass ein konstanter Wert der Péclet-Zahl beim Einsetzen der Konvektion die Ergebnisse besser korreliert als ein konstanter Wert des Verstärkungsfaktors. Theoretische Gründe für dieses Ergebnis werden angegeben.

ВОЗНИКНОВЕНИЕ ЕСТЕСТВЕННОЙ КОНВЕКЦИИ ПРИ ЗАВИСИМЫХ ОТ ВРЕМЕНИ ПРОФИЛЯХ

Аннотация—Проводились эксперименты по возникновению естественной конвекции в глубоких резервуарах (глубина проникновения мала по сравнению с толщиной слоя жидкости). Жидкости в большом диапазоне чисел Прандтля нагревались снизу. Постоянное значение числа Рейля, отнесенное к толщине пограничного слоя (Ra_i), учитывает граничные условия на поверхности и глубину жидкости. Эти эффекты, а также влияние числа Прандтля и расстояния от стенки можно объяснить с помощью теории зависимости между скоростью роста и усмирением; однако было установлено, что данные лучше описываются с помощью постоянного этого числа Пекле при возникновении конвекции, чем с помощью постоянных коэффициентов усиления. Приводится теоретическое обоснование этого результата.